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# QCD-Based Interpretation of the Lepton Spectrum in Inclusive $\bar{B} \rightarrow X_u \ell \bar{\nu}$ Decays

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## Abstract

We present a QCD-based approach to the endpoint region of the lepton spectrum in  $\bar{B} \rightarrow X_u \ell \bar{\nu}$  decays. We introduce a genuinely nonperturbative form factor, the shape function, which describes the fall-off of the spectrum close to the endpoint. The moments of this function are related to forward scattering matrix elements of local, higher-dimension operators. We find that nonperturbative effects are dominant over a finite region in the lepton energy spectrum, the width of which is related to the kinetic energy of the  $b$ -quark inside the  $B$  meson. Applications of our method to the extraction of fundamental standard model parameters, among them  $V_{ub}$ , are discussed in detail.

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# 1 Introduction

Recently, much progress has been achieved in the understanding of inclusive weak decays of hadrons containing a heavy quark  $Q$ . Using the theoretical tools of the operator product expansion and the heavy quark effective theory (HQET) [1, 2, 3, 4, 5], one can construct a systematic expansion of the (differential) decay distributions in powers of  $\Lambda/m_Q$ , where  $\Lambda$  is a characteristic low-energy scale of the strong interactions [6, 7, 8, 9, 10]. Quite remarkably, the parton model emerges as the leading term in this QCD-based expansion, and the nonperturbative corrections to it are suppressed by a factor  $\Lambda^2/m_Q^2$ . The fact that there are no first-order power corrections relies on a particular definition of  $m_Q$ , which is provided in a natural way by requiring that there be no residual mass term for the heavy in HQET [11, 12]. This definition is unique and can be regarded as a nonperturbative generalization of the concept of a pole mass.

The availability of a systematic, QCD-based expansion of inclusive decay rates raises the hope for a better understanding of these processes in general, and in particular for a more reliable extraction of the standard model parameters  $m_b$ ,  $m_c$ ,  $V_{cb}$ , and  $V_{ub}$ , which was so far hindered by strong model dependence. For a determination of  $V_{ub}$ , however, it is essential to understand the endpoint region of the lepton spectrum, which is of genuinely nonperturbative nature. Although the new methods developed in Refs. [7, 8, 9, 10] provide an important step towards this goal, they are not directly applicable to the endpoint region. The difficulties arise from the fact that close to the endpoint the expansion parameter is no longer  $\Lambda/m_b$ , but  $\Lambda/(m_b - 2E_\ell)$ , and thus the theoretical prediction becomes singular when the lepton energy approaches the parton model endpoint  $E_{\ell,\max} = m_b/2$ . It is then not obvious how to interpret the theoretical predictions.

To see what the problem is, consider the theoretical prediction for the lepton spectrum in  $\bar{B} \rightarrow X_u \ell \bar{\nu}$  decays. Including the leading nonperturbative corrections, one obtains [7, 8, 9]

$$\frac{1}{2\Gamma_b} \frac{d\Gamma}{dy} = y F(y) \Theta(1-y) - \frac{\lambda_1 + 33\lambda_2}{6m_b^2} \delta(1-y) - \frac{\lambda_1}{6m_b^2} \delta'(1-y), \quad (1)$$

where

$$y = \frac{2E_\ell}{m_b}, \quad \Gamma_b = \frac{G_F^2 |V_{ub}|^2}{192\pi^3} m_b^5, \quad (2)$$

and

$$F(y) = (3 - 2y)y + \frac{5y^2}{3} \frac{\lambda_1}{m_b^2} + (6 + 5y)y \frac{\lambda_2}{m_b^2}. \quad (3)$$

For simplicity, we do not include perturbative QCD corrections, which have been calculated in Refs. [13, 14]. The hadronic parameters  $\lambda_1$  and  $\lambda_2$  are related to the kinetic energy  $K_b$  of the heavy quark inside the  $B$  meson, and to the mass splitting between  $B$  and  $B^*$  mesons [15]:

$$K_b = -\frac{\lambda_1}{2m_b}, \quad m_{B^*}^2 - m_B^2 = 4\lambda_2. \quad (4)$$

The singular structure of the operator product expansion close to the endpoint at  $y = 1$  manifests itself in the appearance of  $\delta$ -function (and higher) distributions. Certainly, one cannot trust the shape of the theoretical spectrum close to the endpoint. Nevertheless, integrating (1) with a smooth weight function, one obtains well-behaved results for quantities such as the total decay rate and the average lepton energy:

$$\Gamma = \Gamma_b \left\{ 1 + \frac{\lambda_1 - 9\lambda_2}{2m_b^2} \right\}, \quad \langle E_\ell \rangle = \frac{7m_b}{20} \left\{ 1 - \frac{7\lambda_1 + 57\lambda_2}{14m_b^2} \right\}. \quad (5)$$

The coefficients of the singular terms give nonvanishing contributions to these integrated quantities. There is thus some relevant physical information contained in these terms.

Bigi et al. [7] have advocated to integrate over the singularities before confronting the theoretical prediction for the lepton spectrum with data. They argued that one has to integrate over a finite energy interval of at least several hundred MeV, corresponding to a region of order  $1/m_b$  in the variable  $y$ . This proposal is based on the idea of quark-hadron duality, which implies that when one sums over a sufficient number of exclusive hadronic modes, the decay probability into hadrons equals the decay probability into free quarks.<sup>1</sup> Note that the  $\delta$ -function term in (1) contributes to the integrated spectrum, but the term proportional to  $\delta'(1 - y)$  does not. Similarly, more

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<sup>1</sup>One expects that duality holds for the lepton spectrum even in the endpoint region, which extends over an interval of order  $1/m_b$  in  $y$ . Only in a tiny region of order  $1/m_b^2$  below the physical endpoint, the spectrum is dominated by a few exclusive modes.

singular terms, which appear at higher orders in the  $1/m_b$  expansion, do not contribute.

A slightly modified procedure was proposed by Manohar and Wise [9]. They chose to smear the spectrum with a gaussian distribution of width  $\Delta y$ . Empirically, they found that  $\Delta y \sim 0.2 - 0.5$  is necessary to obtain from the theoretical prediction a smooth lepton spectrum, which can be compared to data. This procedure has the disadvantage that the results depend on the smearing function, and that the choice of  $\Delta y$  is ad hoc. When the smearing function is chosen to be symmetric, it again follows that the term proportional to  $\delta'(1 - y)$  does not contribute to the smeared spectrum.

The frustrating conclusion of these analyses is that the new theoretical methods are only of very limited use for a more reliable determination of  $V_{ub}$ , since the region of the lepton spectrum which is accessible to a measurement is smaller than the region over which the theoretical spectrum has to be integrated in order to obtain a reasonable result.

As proposed in Refs. [7, 8, 9], the theoretical description is to a large extent ignorant to the rich physical information contained in the lepton spectrum close to the endpoint. In this paper, we shall suggest a different approach. It is motivated by a very simple observation: In the parton model, the endpoint region of the lepton spectrum is described by a step function, the location of which is determined by the kinematics of a free quark decay. The true physical endpoint, however, is determined by the decay kinematics of hadrons. Hence, when QCD is trying to tell us something about the redistribution of the endpoint region due to nonperturbative effects, it can only do this by the occurrence of singular functions. Our approach will allow us to extract the physical information contained in the singular terms in the QCD-predicted lepton spectrum in a systematic way, and to all orders in the  $1/m_b$  expansion. To this end, we shall introduce the concept of a *shape function*, which is a genuinely nonperturbative form factor that describes the fall-off of the spectrum in the endpoint region. We find that the characteristic width of this region is given by  $\sigma_y = (-\lambda_1/3m_b^2)^{1/2}$ , corresponding to a *finite* region in the lepton energy. Although there do not appear first-order power corrections in (1), it is important to realize that there exists a small region where the true spectrum is very different from the theoretical prediction. This difference is described by the shape function. We will show that the *moments* of this function can be addressed in QCD, and can be related to hadronic parameters (such as  $\lambda_1$  and  $\lambda_2$ ) that are defined in terms of for-

ward scattering matrix elements of local, higher-dimension operators. To all orders in  $1/m_b$ , the leading contributions to the moments can be given in closed form.

We believe that our approach will eventually lead to a better understanding of the nonperturbative aspects of the lepton spectrum close to the endpoint. It establishes the connection between the experimentally observed lepton spectrum and the underlying theory of QCD. This connection works in both directions: Theoretical ideas about the moments of the shape function can help to analyze the lepton spectrum and to determine the values of the quark masses and mixing angles. On the other hand, from a precise measurement of the spectrum in the endpoint region one can extract the shape function and with it some fundamental matrix elements of higher-dimension operators in QCD.

Starting from a resummation of the theoretical lepton spectrum, we present in Sec. 2 a heuristic argument that leads to the notion of a function  $S(y)$ , which describes the fall-off of the spectrum in the endpoint region. In Sec. 3, we introduce this shape function, discuss its properties, and derive relations for the first two moments of  $S(y)$ . Sec. 4 is devoted to a formal definition of the shape function to all orders in  $1/m_b$ . The leading contributions to the moments are related to forward scattering matrix elements of local, higher-dimension operators in HQET. For the purpose of illustration, a simple model calculation of the shape function is presented in Sec. 5. In Sec. 6, we summarize our results, indicate possible further applications and improvements of the method, and give some conclusions.

## 2 Resummation of the singular terms

To motivate the concept of a shape function, let us try to resum the singular contributions in (1) into a corrected parton model decay distribution. Obviously, the term proportional to  $\delta(1 - y)$  can be absorbed by a shift of the argument of the step function in the leading-order term. More interesting is the contribution proportional to  $\delta'(1 - y)$ . It arises at second order in the expansion of the step function. However, since there is no  $\delta$ -function contribution of first order in  $1/m_b$ , it follows that one needs more than one step function, resulting in a *dispersion* of the spectrum. In fact, to order  $1/m_b^2$ ,

we can rewrite the theoretical spectrum in the following way:

$$\frac{1}{2\Gamma_b} \frac{d\Gamma}{dy} = y F(y) \frac{1}{N} \sum_{i=1}^N \Theta(1 - y + \varepsilon_i), \quad (6)$$

where

$$\delta y = \frac{1}{N} \sum_{i=1}^N \varepsilon_i = -\frac{\lambda_1 + 33\lambda_2}{6m_b^2}, \quad \sigma_y^2 = \frac{1}{N} \sum_{i=1}^N \varepsilon_i^2 = -\frac{\lambda_1}{3m_b^2}. \quad (7)$$

From the second relation it follows that the displacements  $\varepsilon_i$  are of order  $1/m_b$ . Thus, the first relation corresponds to a nontrivial cancellation. Note that  $\sigma_y$  can be identified with the characteristic width of the endpoint region, i.e., the region over which the spectrum is dominated by nonperturbative effects. This is already a remarkable conclusion: The coefficient of the most singular term in (1), which had no effect in the approaches of Refs. [7, 9], determines the size of the endpoint region.

At this point, it is worthwhile to obtain some estimates of the nonperturbative corrections. From the observed value of the mass splitting between  $B$  and  $B^*$  mesons one obtains  $\lambda_2 \simeq 0.12 \text{ GeV}^2$ . The parameter  $\lambda_1$ , on the other hand, is not directly related to an observable. Recently, we have shown that the field-theory analog of the virial theorem relates the kinetic energy of a heavy quark inside a hadron (and thus  $\lambda_1$ ) to a matrix element of the gluon field strength tensor [16]. This theorem makes explicit an intrinsic “smallness” of  $\lambda_1$ , which was not taken into account in existing QCD sum rule calculations of this parameter [17, 18, 19]. As a consequence, we expect that  $(-\lambda_1)$  is considerably smaller than predicted in these analyses. Here we shall use the range  $-\lambda_1 = 0.05 - 0.30 \text{ GeV}^2$ . According to its definition,  $\lambda_1$  is negative, so that the width  $\sigma_y$  in (7) is well defined. Using these numbers, as well as  $m_b = 4.8 \text{ GeV}$ , we estimate  $\delta y \simeq -0.03$  and  $\sigma_y \simeq 0.03 - 0.07$ . We can multiply these quantities by  $m_b/2$  to obtain the corresponding shift and spread in the lepton energy spectrum. They are  $\delta E \simeq -65 \text{ MeV}$  and  $\sigma_E = (-\lambda_1/12)^{1/2} \simeq 65 - 160 \text{ MeV}$ . The value of  $\sigma_E$  can be compared to the width of the gap between the parton model endpoint of the spectrum and the physical endpoint, which, if we neglect the pion mass, is located at  $E_{\ell, \text{max}}^{\text{phys}} = m_B/2$ . This width is  $\Delta E \simeq (m_B - m_b)/2 \simeq 240 \text{ MeV}$ . These numbers seem quite reasonable. In fact, assuming that the distribution of the displacements  $\varepsilon_i$  around  $y = 1$  is approximately symmetric, we have to

require that  $\sigma_E < \Delta E$ , which is equivalent to  $-\lambda_1 < 3(m_B - m_b)^2$ . For reasonable values of  $\lambda_1$ , this bound is always satisfied.

In Fig. 1, we show the resummed lepton energy spectrum (6) for  $\lambda_1 = -0.2 \text{ GeV}^2$ ,  $\lambda_2 = 0.12 \text{ GeV}^2$ ,  $N = 10$ , and a particular set of  $\varepsilon_i$  satisfying the constraints in (7). For this choice of parameters, the dispersion in the spectrum is such that the endpoint falls close to the physical endpoint at  $y \simeq 1.1$ . Our reinterpretation of the QCD-predicted lepton spectrum has led to a reasonable shape which, however, is quite arbitrary. In fact, increasing  $N$ , we can generate any decreasing function that satisfies the constraints in (7). In the following section, we will introduce a shape function  $S(y)$  instead of the sum over step functions. The constraints will then turn into predictions for the first two moments of this function.

### 3 The shape function

We proceed by replacing the sum over step functions in (6) by a continuous function  $\vartheta(y)$ , which we furthermore decompose as  $\vartheta(y) = \Theta(1 - y) + S(y) F(1)/F(y)$ . The form of the second term is chosen for later convenience. We shall refer to  $S(y)$  as the shape function. The support of this function is restricted to a small interval of width  $2\Delta$  around  $y = 1$ , where  $\Delta$  of order  $1/m_b$ .<sup>2</sup> Some basic properties of  $S(y)$  can be derived from the physical requirements that the differential decay rate be positive, and that  $\vartheta(y)$  be a continuous function. We note that

$$\begin{aligned} S(y) &= 0 \quad \text{if } |y - 1| > \Delta, \\ S(y) &\geq 0 \quad \text{if } y \geq 1, \\ \lim_{\epsilon \rightarrow 0} S(1 + \epsilon) - S(1 - \epsilon) &= 1, \\ \lim_{\epsilon \rightarrow 0} S'(1 + \epsilon) - S'(1 - \epsilon) &= 0. \end{aligned} \tag{8}$$

Moreover, we expect that  $S'(y) \leq 0$  if  $y \neq 1$ , but we shall not impose this as a condition on  $S(y)$ .

As emphasized in Refs. [7, 9], because of the singular form of the operator product expansion one has to integrate the theoretical lepton spectrum with

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<sup>2</sup>More precisely, we should require that  $S(y)$  be exponentially small outside an interval of width  $2\Delta$ .

a smooth function before it can be compared to data. On the set of smooth functions (i.e., functions of  $y$  which are slowly varying over scales of order  $1/m_b$ ), a rapidly varying function such as  $S(y)$ , which vanishes outside a small interval around  $y = 1$ , obeys a singular expansion of the form<sup>3</sup>

$$S(y) = \sum_{n=0}^{\infty} \frac{M_n}{n!} \delta^{(n)}(1-y), \quad (9)$$

where the *moments*  $M_n$  are defined as

$$M_n = \int_0^{\infty} dy (y-1)^n S(y) = \int_{-\Delta}^{\Delta} dy (y-1)^n S(y). \quad (10)$$

To see that (9) is correct, assume that any reasonable test function  $f(y)$  can be Taylor-expanded around  $y = 1$ , with the result that

$$\int_0^{\infty} dy f(y) S(y) = \sum_{n=0}^{\infty} \frac{M_n}{n!} f^{(n)}(1). \quad (11)$$

In terms of the shape function, the theoretical lepton spectrum takes the form

$$\frac{1}{2\Gamma_b} \frac{d\Gamma}{dy} = y \left[ F(y) \Theta(y) + F(1) S(y) \right]. \quad (12)$$

From a comparison with (1), we find that the first two moments of  $S(y)$  must satisfy<sup>4</sup>

$$\begin{aligned} M_0 &= -\frac{11\lambda_2}{2m_b^2} \simeq -2.9\%, \\ M_1 &= -\frac{\lambda_1}{6m_b^2} = \frac{\sigma_y^2}{2} \simeq (0.4 - 2.2) \times 10^{-3}. \end{aligned} \quad (13)$$

Notice that  $M_n = \mathcal{O}(1/m_b^{n+1})$  by dimensional analysis. Hence, the QCD prediction that  $M_0 = \mathcal{O}(1/m_b^2)$  corresponds to a nontrivial cancellation: The

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<sup>3</sup>This procedure is familiar from the multipole expansion of a localized distribution of charges in electrodynamics.

<sup>4</sup>Our judicious choice of the prefactor  $y$  in front of  $S(y)$  has eliminated the contribution of  $\lambda_1$  to  $M_0$ .



area under the shape function (almost) vanishes. The first moment, which is related to the characteristic width of the endpoint region, is of the expected order of magnitude.

The concept of a shape function exploits to full extent the physical information contained in the coefficients of the singular terms in the QCD-predicted lepton spectrum. We find that over a region of width  $2\Delta \propto 1/m_b$ , the spectrum is of genuinely nonperturbative nature and described by a function  $S(y)$ , the moments of which can be addressed in QCD. When one goes to higher orders in the  $1/m_b$  expansion, one can address higher moments. In fact, the moments obey an expansion of the form

$$M_n = \frac{a_n}{m_b^{n+1}} + \frac{b_n}{m_b^{n+2}} + \dots, \quad (14)$$

where so far we know the coefficients  $a_0 = 0$ ,  $b_0 = -11\lambda_2/2$ , and  $a_1 = -\lambda_1/6$ . With the exception of the moment  $M_0$ , where the leading term  $a_0$  vanishes, we may argue that it would be a good approximation to know the leading coefficient  $a_n$  for each moment. The corrections involving  $b_n$  only change the moments by small amounts. On the other hand, knowledge of a new moment teaches us a new piece of essential information about the shape of the spectrum in the endpoint region. The higher moments give a small contribution to integrated quantities such as the total decay rate, simply because in (10) one integrates over a small region. Nevertheless, they can affect the shape of the endpoint region in a substantial way. What is relevant to the shape are the rescaled moments

$$\mathcal{M}_n = \int_0^\infty dE_\ell (2E_\ell - m_b)^n S(E_\ell) = m_b^{n+1} M_n = a_n + \frac{b_n}{m_b} + \dots, \quad (15)$$

which remain nonzero in the limit  $m_b \rightarrow \infty$ . As an illustration of the importance of higher moments, we show in Fig. 2 two shape functions which have the same first two moments  $M_0$  and  $M_1$ , but different third (and higher) moments. The total decay rate and the average lepton energy are the same in both cases (up to terms of order  $1/m_b^3$ ), but obviously the behavior close to the endpoint is quite different.

## 4 Formal definition of the shape function

The above discussion shows that for an understanding of the lepton spectrum in the endpoint region, it is insufficient to truncate the theoretical calculation at order  $1/m_b^2$ . Instead, what one needs to investigate to all orders in  $1/m_b$  are the coefficients  $a_n$  in (14) and (15). They arise from the most singular terms in the shape function. In this section, we give a formal definition of these terms to all orders in  $1/m_b$ . This will provide us with a relation between the coefficients  $a_n$  and forward scattering matrix elements of local, higher-dimension operators in HQET.

As mentioned in the introduction, the derivation of the lepton spectrum is based on the operator product expansion in connection with an expansion of hadronic matrix element in powers of  $1/m_b$ , as provided by HQET. This is explained in detail in Refs. [7, 8, 9, 10]. Using the same technology, we can derive a closed expression for the most singular terms of the shape function, where  $S(y)$  is defined as in (12) as the sum of all terms in the theoretical spectrum that become singular in the limit  $y \rightarrow 1$ . We obtain the formal result

$$S(y) = \left\langle \Theta \left[ 1 - y + \frac{2}{m_b} (v - \hat{p}) \cdot iD \right] - \Theta(1 - y) \right\rangle + \text{less singular terms}, \quad (16)$$

which is valid to all orders in the  $1/m_b$  expansion. Here

$\hat{p} = p_\ell/m_b$  denotes the rescaled lepton momentum, and we define the expectation value of an operator  $O$  as

$$\langle O \rangle = \frac{\langle B(v) | \bar{h}_v O h_v | B(v) \rangle}{\langle B(v) | \bar{h}_v h_v | B(v) \rangle}. \quad (17)$$

Here,  $h_v$  is the velocity-dependent heavy quark field in HQET [2], and the states are the eigenstates of the corresponding effective Lagrangian. Details of the derivation of (16), as well as the extension to the case of  $\bar{B} \rightarrow X_c \ell \bar{\nu}$  decays, will be given elsewhere [20].

Expanding our result in powers of  $1/m_b$ , we obtain

$$S(y) = \sum_{n=1}^{\infty} \frac{1}{n!} \delta^{(n-1)}(1-y) \left( \frac{2}{m_b} \right)^n (v - \hat{p})_{\mu_1} \dots (v - \hat{p})_{\mu_n} \langle iD^{\mu_1} \dots iD^{\mu_n} \rangle + \text{less singular terms}. \quad (18)$$

The forward scattering matrix elements between  $B$  mesons, or between any other hadronic states that are unpolarized, can be parameterized in the form

$$\langle iD^{\mu_1} \dots iD^{\mu_n} \rangle = A_n v^{\mu_1} \dots v^{\mu_n} + \text{terms with } g^{\mu_i \mu_j}. \quad (19)$$

Since  $(v - \hat{p})^2 = (1 - y)$  vanishes at the endpoint, only the coefficients  $A_n$  contribute to the most singular terms in  $S(y)$ . For the same reason, we can replace factors  $2v \cdot (v - \hat{p}) = (2 - y)$ , which arise upon contraction of the indices in (18), by 1. This leads to the following expression for the shape function:

$$S(y) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{A_n}{m_b^n} \delta^{(n-1)}(1 - y) + \text{less singular terms}. \quad (20)$$

From a comparison with (9) and (14), we obtain for the moments of  $S(y)$ :

$$M_n = \frac{1}{(n+1)} \frac{A_{n+1}}{m_b^{n+1}}, \quad a_n = \frac{A_{n+1}}{n+1}. \quad (21)$$

The first three coefficients  $A_n$  are given by  $A_0 = 1$ ,  $A_1 = 0$ , and  $A_2 = -\lambda_1/3$ .

## 5 A model calculation

It is instructive to consider a simple model for the shape function  $S(y)$ . For this purpose, we evaluate the expectation value in (16) following the phenomenological approach of Altarelli et al. (ACM) [14]. We emphasize, however, that this is mainly meant as an illustrative example rather than a prediction of the physical shape function. In fact, we will see very clearly the limitations and shortcomings of the ACM approach.

In the ACM model, one assumes the validity of the parton model and incorporates bound state effect by assigning a momentum distribution  $\phi(|\vec{p}_b|)$  to the heavy quark inside the  $B$  meson at rest. It is then appropriate to replace the covariant derivative in (16) by the spatial components of the heavy quark momentum  $\vec{p}_b$ . The gluon field in the covariant derivative is neglected. Accordingly, in the rest frame of the  $B$  meson, one makes the replacement

$$\frac{2}{m_b} (v - \hat{p}) \cdot iD \rightarrow \frac{2\vec{p}_\ell \cdot \vec{p}_b}{m_b^2} = \frac{y |\vec{p}_b|}{m_b} \cos \vartheta, \quad (22)$$

where  $\vartheta$  is the angle between the lepton and the heavy quark momentum. Since we are interested in the behavior in the endpoint region, we can set  $y = 1$ . The matrix element in (16) is now replaced by an integral over the momentum distribution of the heavy quark:

$$S_{\text{ACM}}(y) = \int_0^\infty d|\vec{p}_b| |\vec{p}_b|^2 \phi(|\vec{p}_b|) \int_{-1}^1 \frac{d \cos \vartheta}{2} \left\{ \Theta \left[ 1 - y + \frac{|\vec{p}_b|}{m_b} \cos \vartheta \right] - \Theta(1 - y) \right\}. \quad (23)$$

It is straightforward to calculate the moments of this model shape function, and from (21) the corresponding predictions for the hadronic matrix elements  $A_n$ . We find that  $A_{2n+1}^{\text{ACM}} = 0$ , and

$$A_{2n}^{\text{ACM}} = \langle |\vec{p}_b|^{2n} \rangle = \int_0^\infty d|\vec{p}_b| |\vec{p}_b|^{2(n+1)} \phi(|\vec{p}_b|). \quad (24)$$

In the ACM model, one assumes a gaussian distribution,

$$\phi(|\vec{p}_b|) = \frac{4}{\sqrt{\pi} p_F^3} \exp \left( - \frac{|\vec{p}_b|^2}{p_F^2} \right), \quad (25)$$

where  $p_F$  is the Fermi momentum. This leads to the shape function

$$S_{\text{ACM}}(y) = \left[ \frac{1}{2} - \Theta(1 - y) \right] \Phi \left( \frac{m_b}{p_F} |y - 1| \right). \quad (26)$$

$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dt e^{-t^2}$  denotes the complementary error function. For the coefficients  $A_{2n}$ , one obtains

$$A_{2n}^{\text{ACM}} = \frac{(2n-1)!!}{2^n} p_F^{2n}. \quad (27)$$

Comparing this to the general relation  $A_2 = -\lambda_1/3$ , we derive

$$-\lambda_1^{\text{ACM}} = \frac{3}{2} p_F^2 \simeq 0.08 \text{ GeV}^2, \quad (28)$$

where we have used  $p_F \simeq 230 \text{ MeV}$  as obtained from the most recent fit of the ACM model to experimental data [21]. The model thus predicts a rather small value of  $-\lambda_1$ .

In Fig. 3, we show the shape function of the ACM model for three different values of the Fermi momentum. We stress again that this simple model calculation is presented for pedagogical purposes only. In particular, note that in the ACM model the even moments of the shape function (corresponding to odd coefficients  $A_{2n+1}$ ) vanish by rotational invariance. This is a consequence of the fact that one replaces the *operator* of the covariant derivative by a *c-number* momentum vector. Whereas in QCD the commutator of two covariant derivatives gives the gluon field strength tensor, in the ACM model this commutator vanishes. However, exactly those terms involving the gluon field are responsible for an asymmetry in the shape function around  $y = 1$ . To see this, consider the matrix element of three covariant derivatives. Using the equations of motion of HQET, it is easy to show that

$$\langle iD^\mu iD^\nu iD^\alpha \rangle = A_3 (v^\mu v^\alpha - g^{\mu\alpha}) v^\nu, \quad (29)$$

and taking the antisymmetric combination in  $\mu$  and  $\nu$ , we find that  $A_3$  is related to a matrix element involving the gluon field strength tensor:

$$\langle ig_s G^{\mu\nu} iD^\alpha \rangle = A_3 (v^\mu g^{\nu\alpha} - v^\nu g^{\mu\alpha}). \quad (30)$$

In QCD, there is no reason why such a matrix element should vanish. Hence, we expect an asymmetry of the physical shape function. The above example is instructive since it shows that a measurement of the moments of the shape function can provide quite fundamental information about the dynamical properties of the theory of strong interactions.

## 6 Summary and conclusions

We have presented a QCD-based approach to the inclusive lepton energy spectrum in  $\bar{B} \rightarrow X_u \ell \bar{\nu}$  decays. We have introduced the concept of a shape function, which is a genuinely nonperturbative object that describes the rapid fall-off of the spectrum in the endpoint region. The moments of this function obey a very simple relation to forward scattering matrix elements of local, higher-dimension operators. QCD predicts that the leading contribution to the first moment vanishes. The second moment, which is a measure of the size of the endpoint region, is proportional to the expectation value of the kinetic energy of the heavy quark inside the hadron.

Our approach goes beyond previous work on the lepton spectrum [7, 8, 9], which was applicable only for lepton energies not too close to the endpoint. It aims at a systematic use of the rich source of information contained in the endpoint region. Whereas the main part of the spectrum is determined by kinematics and only receives small nonperturbative corrections, the behavior close to the endpoint is characterized by an infinite set of hadronic matrix elements. It is worth noting that, although there are no first-order power corrections to the spectrum and decay rate at small lepton energies, there exists a small region where the true spectrum is very different from the theoretical prediction.

There are obvious improvements of the analysis presented here. Before confronting our results with data, it is necessary to include radiative corrections. For the case of  $\bar{B} \rightarrow X_u \ell \bar{\nu}$  decays, they are known to affect the parton model spectrum in a significant way [13, 14]. Such corrections will affect the form of the shape function, too. We expect small perturbative corrections of order  $\alpha_s(m_b)$  to all moments of the shape function. Moreover, radiative corrections will wash out the step in the shape function at  $y = 1$ , resulting in a rapidly varying, but not discontinuous, behavior. Another important generalization of our approach is that to the case of  $\bar{B} \rightarrow X_c \ell \bar{\nu}$  decays. The nonvanishing mass of the charm quark will lead to technical complications, but conceptually there is no problem in defining a shape function  $S(y, \rho)$  for any value of  $\rho = m_c^2/m_b^2$ . The (appropriately defined) moments of this generalized shape function are still related to the same hadronic matrix elements as in the case of a massless final state quark. Work on all these issues is in progress and will be reported elsewhere [20].

We believe that our approach will eventually lead to a better understanding of the nonperturbative aspects of inclusive decay spectra, with implications for the measurement of some of the fundamental parameters of the standard model, such as the heavy quark masses and the elements  $V_{cb}$  and  $V_{ub}$  of the Kobayashi-Maskawa matrix. Currently, the most promising applications of the method seem to be the following:

In  $B$  decays into charmless final states, an understanding of the endpoint region is crucial for a reliable determination of  $V_{ub}$ . The approach of Refs. [7, 8, 9] cannot be used for this purpose, since current measurements are limited to a small energy range  $2.3 \leq E_\ell \leq 2.6$  GeV (corresponding to  $0.96 \leq y \leq 1.08$  for  $m_b = 4.8$  GeV), which is too close to the endpoint. What is needed is some insight into the nonperturbative effects relevant to

the shape of the spectrum in the endpoint region. An expansion in powers of  $1/m_b$  is not suitable for such a situation. The relevant physics is encoded in the moments of the shape function, which are related to forward scattering matrix elements of local, higher-dimension operators. Such matrix elements can be addressed using nonperturbative techniques such as lattice gauge theory or QCD sum rules. It may even be possible in these approaches to attempt a direct calculation of the shape function from its definition in (16).

For  $\bar{B} \rightarrow X_c \ell \bar{\nu}$  transitions, the situation is very different. Already, there exist very accurate measurements of the lepton spectrum in this case. For a determination of  $V_{cb}$  and of the quark masses  $m_b$  and  $m_c$ , an understanding of the endpoint region is thus not a necessary requirement. However, these decays offer the exciting possibility to extract the shape function from the data, simply by subtracting the (corrected) parton model spectrum from the measured distribution. One could then compute the first few moments of the shape function and extract some of the coefficients  $A_n$ , which encode fundamental dynamical properties of QCD. The fact that QCD predicts the size of the first moment  $M_0$  in (13) provides an important constraint, which can help to obtain a very precise determination of the  $b$ -quark mass.

We conclude with a speculation about yet another exciting possibility, namely to combine the analyses of  $\bar{B} \rightarrow X_c \ell \bar{\nu}$  and  $\bar{B} \rightarrow X_u \ell \bar{\nu}$  decays. In fact, one can derive a simple relation between the shape functions for these two processes [20]. One can imagine to measure the shape function in  $B$  decays into charmed particles, and then predict the shape function for charmless transitions. This avenue may well be the most promising one with respect to a precise extraction of  $V_{ub}$ .

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## Figure Captions

Figure 1: An example of a resummed lepton energy spectrum according to (6). On the vertical axis, we show  $d\Gamma/dy$  in units of  $2\Gamma_b$ .

Figure 2: Two shape functions with identical moments  $M_0$  and  $M_1$  (a), and the corresponding lepton spectra (b).

Figure 3: The shape function of the ACM model (a) and the corresponding lepton spectrum (b) for  $-\lambda_1 = 0.05 \text{ GeV}^2$  (dashed),  $0.1 \text{ GeV}^2$  (solid), and  $0.2 \text{ GeV}^2$  (dotted). The corresponding values of the Fermi momentum are  $p_F \simeq 180 \text{ MeV}$ ,  $260 \text{ MeV}$ , and  $365 \text{ MeV}$ , respectively. In (b), the parton model spectrum is shown as a grey line.

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